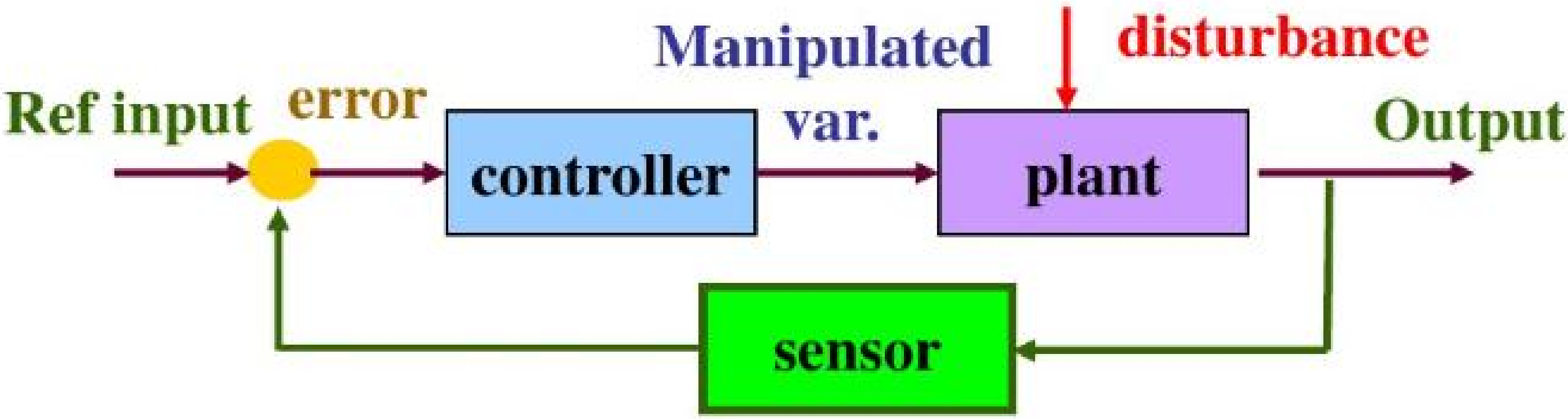
CHAPTER 3

Mathematical Models of Control Systems

How to analyze and design a control system



- The first thing is to establish system model (mathematical model)

Outline

3.1 Differential Equations of Linear Components and Linear Systems

3.2 Complex Domain Mathematical Models of Control Systems

3.3 Block Diagrams of Control Systems

System model

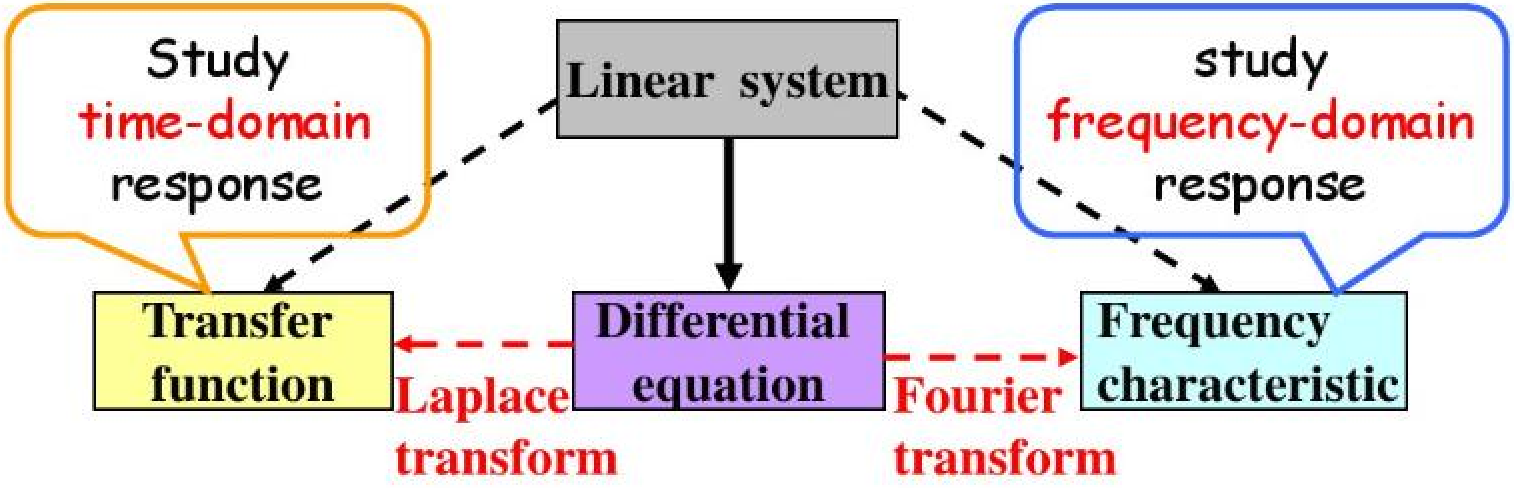
Definition:

- Mathematical expression of dynamic relationship between input and output of a control system.
- Mathematical model is foundation to analyze and design automatic control systems.
- No mathematical model of a physical system is exact. We generally strive to develop a model that is adequate for the problem at hand but without making the model overly complex.

System model

Three models

- Differential equation
- Transfer function
- Frequency characteristic



Modeling methods

Analytic method

解析方法/机理方法

According to

A.Newton's Law of Motion

B.Law of Kirchhoff

C.System structure and parameters

the mathematical expression of system input and output can be derived.

Thus, we build the mathematical model (suitable for simple systems).

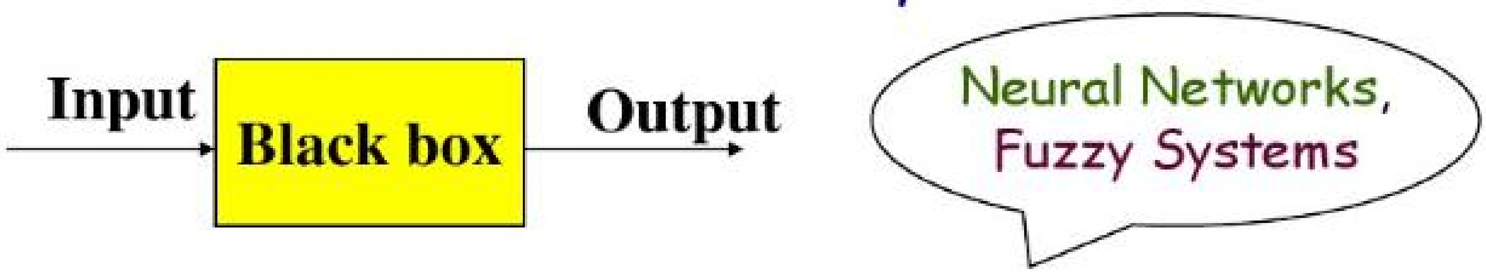
Modeling methods

System identification method

系统辨识方法

Building the system model based on the system input—output signal

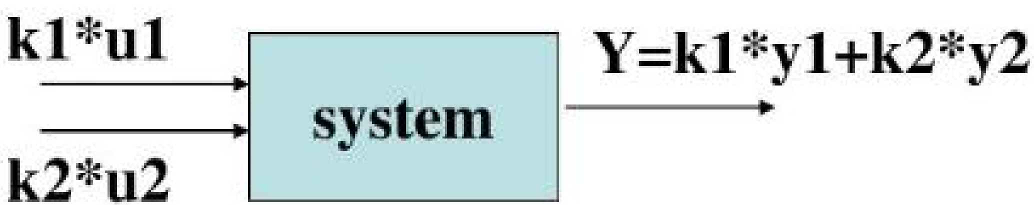
This method is usually applied when there are little information available for the system.



Black box: the system is totally unknown.

Grey box: the system is partially known.

- What is linear system?
  - A system is called linear if the principle of superposition (叠加原理) applies



Is y(t)=u(t)+2 a linear system?

- What is time-invariant system?
  - A system is called time-invariant if the parameters are stationary with respect to time during system operation
  - 时变系统的例子？

Establishment of differential equation

Differential equation

- Linear ordinary differential equations

-- A wide range of systems in engineering are modeled mathematically by differential equations

-- In general, the differential equation of an n-th order system is written

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_1 \frac{dc(t)}{dt} + a_0 c(t) =$$

$$b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_1 \frac{dr(t)}{dt} + b_0 r(t)$$

Establishment of differential equation

- How to establish differential equation

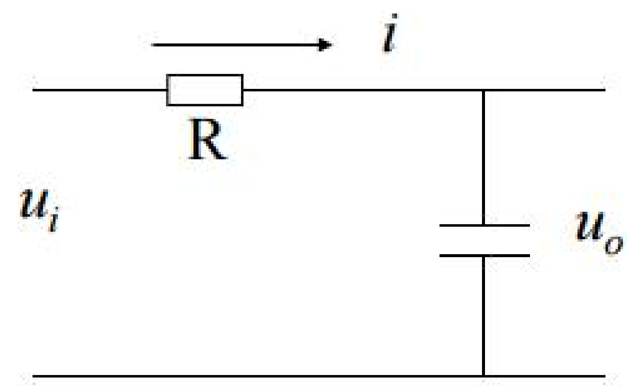
a) Determine the inputs and outputs of the system or component;
b) List differential equations according to the physical rules of each component;
c) Obtain the differential equation sets by eliminating the intermediate variables;
d) Get the overall input-output differential equation of control system.

Establishment of differential equation

Examples 1 RC circuit

According to Kirchhoff's voltage law

$$u_i(t) = Ri(t) + u_o(t)$$



$$i(t)=C\frac{du_o(t)}{dt}$$

$$u_i(t) = RC\frac{du_o(t)}{dt} + u_o(t)$$

It is re-written as in standard form

$$RC\frac{du_o(t)}{dt} + u_o(t)=u_i(t)$$

Establishment of differential equation

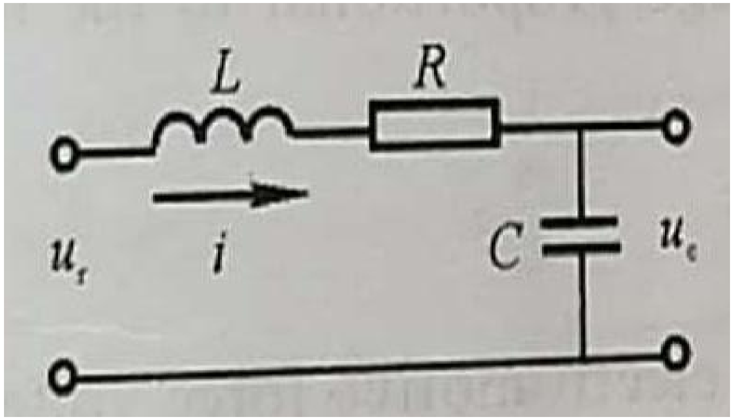
Examples 2 RLC circuit

According to Kirchhoff's voltage law

$$u_r(t) = Ri(t) + L\frac{di(t)}{dt} + u_c$$

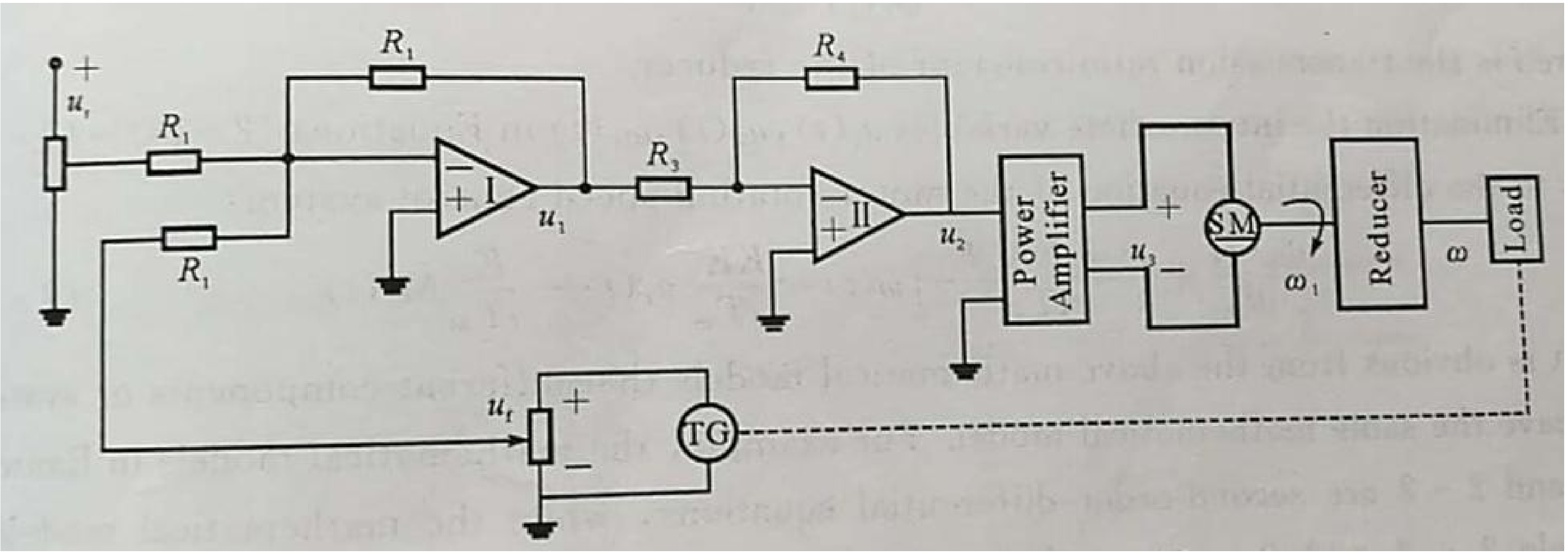
$$i(t)=C\frac{du_c(t)}{dt}$$

$$LC\frac{d^2u_c(t)}{dt^2} + RC\frac{du_c(t)}{dt} + u_c(t) = u_r(t)$$



Establishment of differential equation

Examples 3 DC motor



Establishment of differential equation

Examples 3 DC motor

(1) $u_f(t) = \mathrm{K}_t\,\omega(t)$

(2) $u_e(t) = u_r(t) - u_f(t)$

(3) $u_1(t) = \dfrac{R_2}{R_1} u_e(t)$

(4) $u_2(t) = \dfrac{R_4}{R_3} u_1(t)$

(5) $u_a(t) = K_3 u_2(t)$

(6) $u_a(t) = K u_e(t)$

(7) $T_m \dfrac{d\omega_1(t)}{dt} + \omega_1(t) = K_a u_a(t) - K_c M_c(t)$

(8) $\dfrac{\omega(t)}{\omega_1(t)} = \dfrac{1}{i}$

We have $\dfrac{d\omega(t)}{dt} + \left(\dfrac{i + KK_aK_t}{iT_m}\right)\omega(t) = \dfrac{KK_a}{iT_m} u_r(t) - \dfrac{K_c}{iT_m} M_c(t)$

Why need Laplace transform?

Solving Differential Equations

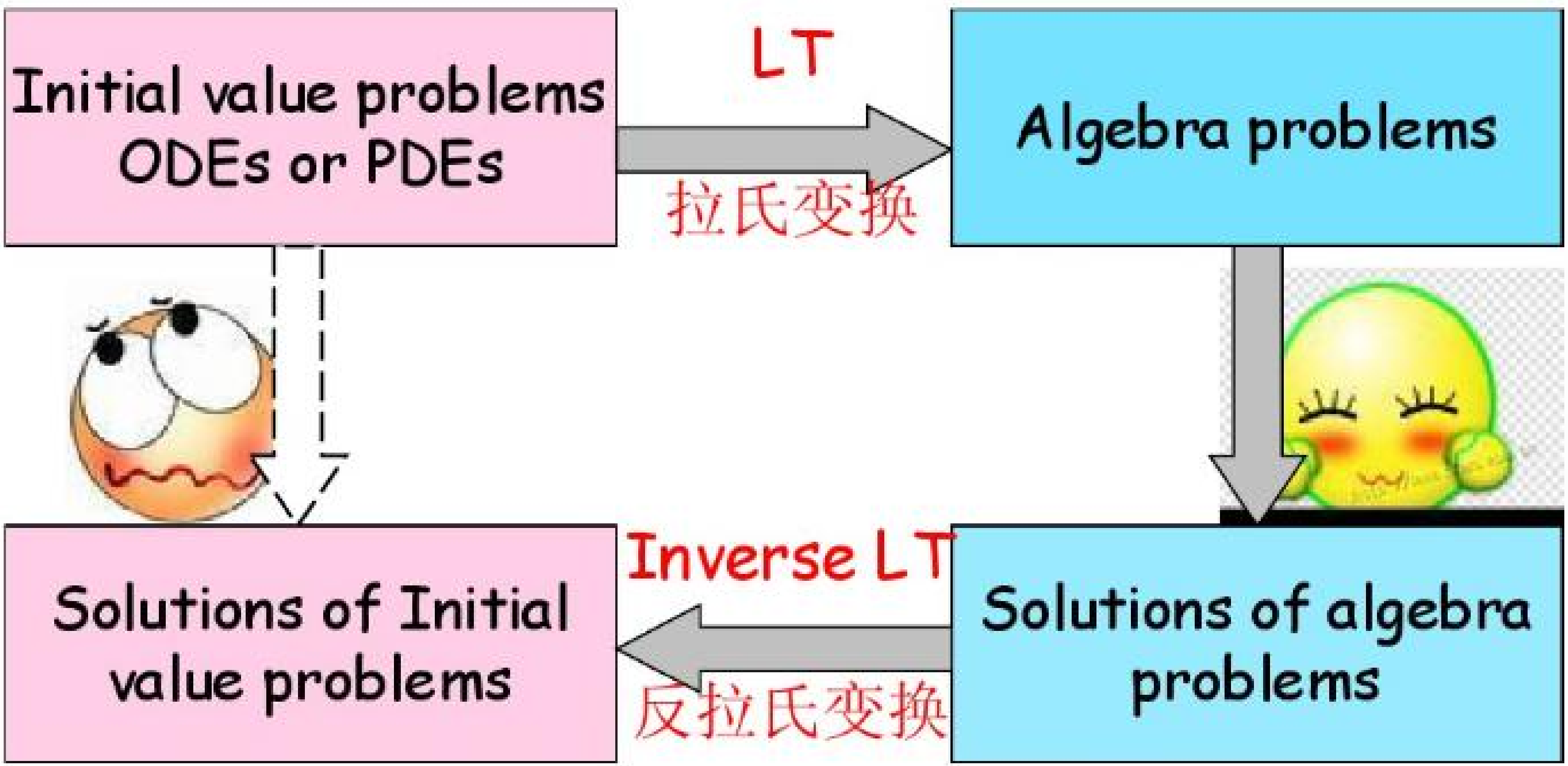
例

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2x + 1$$

Solving linear differential equations with constant coefficients,

- To find the general homogeneous solution （奇次通解） (involving solving the characteristic equation特征方程)
- To find a particular solution of the complete nonhomogeneous equation（非奇次特解） (involving constructing the family of a function)
- To solve the initial value problem （初值问题）

Why need Laplace transform?

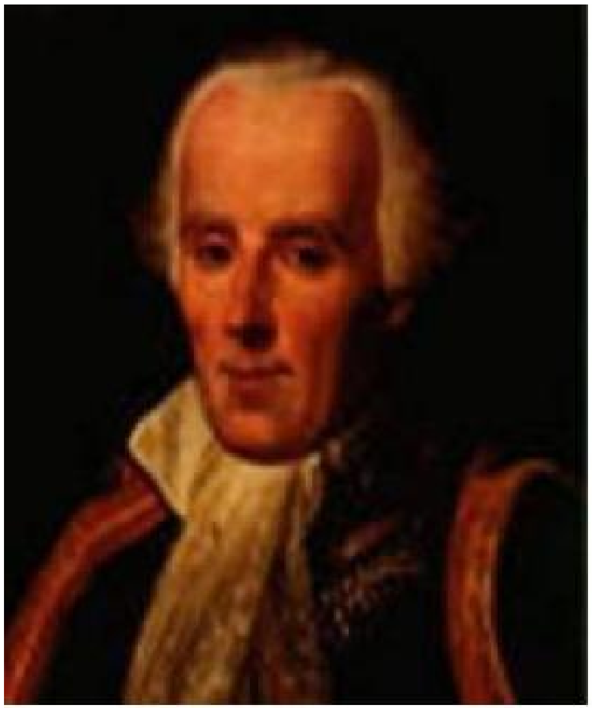


Laplace transform

The Laplace transform of a function f(t) is defined as

$$F(s) = L\left[f(t)\right]$$

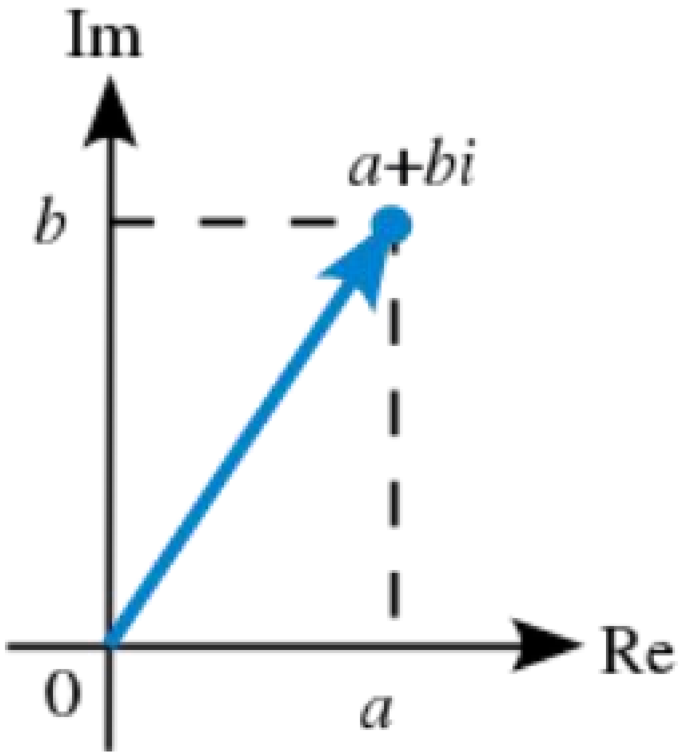
$$= \int_0^\infty f(t) e^{-st} dt$$



Laplace, Pierre-Simon
1749-1827

Laplace transform

- A complex number can be visually represented as a pair of numbers (a, b) forming a vector on a diagram called an Argand diagram, representing the complex plane. "Re" is the real axis, "Im" is the imaginary axis, and i satisfies $i^2 = -1$.



Laplace transform

阶跃信号 $f(t)=A$

$$F(s)=\int_0^\infty f(t)e^{-st}dt=\int_0^\infty Ae^{-st}dt=-\frac{A}{s}e^{-st}\Big|_0^\infty=\frac{A}{s}$$

脉冲信号 $f(t)=\sigma(t)$

$$F(s)=\int_0^\infty f(t)e^{-st}dt=\int_0^\infty \delta(t)e^{-st}dt=1$$

指数信号 $f(t)=e^{-at}$

$$F(s)=\int_0^\infty e^{-at}e^{-st}dt=-\frac{1}{s+a}e^{-(a+s)t}\Big|_0^\infty=\frac{1}{s+a}$$

Laplace transform

f(t)	F(s)	f(t)	F(s)
δ (t)	1	sinwt	$\frac{w}{s^2+w^2}$
1(t)	1/s	coswt	$\frac{s}{s^2+w^2}$
t	$1/s^2$	$e^{-at}\sin wt$	$\frac{w}{(s+a)^2+w^2}$
e^{-at}	1/(s+a)	$e^{-at}\cos wt$	$\frac{s+a}{(s+a)^2+w^2}$

Properties of Laplace transform

(1) Linearity

$$L[af_1(t)+bf_2(t)]=aL[f_1(t)]+bL[f_2(t)]$$

(2) Differentiation

$$L[f'(t)]=sF(s)-f(0)$$

f(0) is the initial value of f(t).

$$L[f^n(t)]=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-f^{n-1}(0)$$

Properties of Laplace transform

(3) Integration

$$L\left[\int f(t)dt\right]=\frac{F(s)}{s}+\frac{f^{-1}(0)}{s}$$

证明（用分部积分）

where $f^{-1}(0)$ is the initial value of integration $\int f(t)dt$

$$L\left[\iint f(t)dt^2\right]=\frac{1}{s^2}F(s)+\frac{1}{s^2}f^{(-1)}(0)+\frac{1}{s}f^{(-2)}(0)$$

Inverse Laplace transform

Definition: Inverse Laplace transform, denoted by $L^{-1}[F(s)]$ is given by

$$f(t)=L^{-1}[F(s)]=\frac{1}{2\pi\cdot j}\int_{C-j\infty}^{C+j\infty}F(s)e^{st}ds(t>0)$$

where C is a real constant。

Note: The inverse Laplace transform operation involving rational functions can be carried out using Laplace transform table and partial-fraction expansion.

Transfer function

Differential equation:

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_1 \frac{dc(t)}{dt} + a_0 c(t) =$$

$$b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_1 \frac{dr(t)}{dt} + b_0 r(t)$$

Laplace Transform:

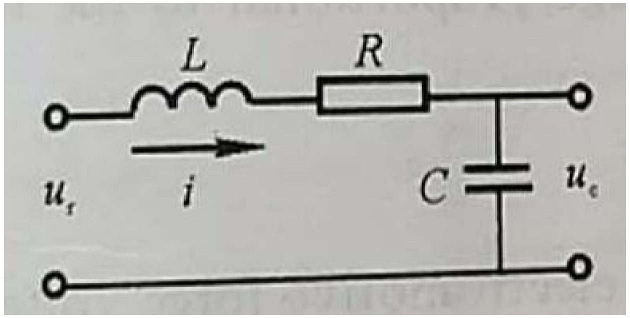
$$\left[a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0\right] C(s)$$

$$= \left[b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0\right] R(s)$$

Transfer function:

$$\frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = G(s)$$

Transfer function of typical component



Example RLC

- Differential equation:

$$LC\frac{d^2u_c(t)}{dt^2}+RC\frac{du_c(t)}{dt}+u_c(t)=u_r(t)$$

- When initial conditions are zero, the transfer function:

$$G(s)=\frac{U_c(s)}{U_r(s)}=\frac{1}{LCs^2+RCs+1}$$

Transfer function

Transfer function:

- A transfer function is defined only for a linear time-invariant system.
- A transfer function between an input and output of a system is defined as the ratio of the Laplace transform of the output to the Laplace transform of the input. Meanwhile, all initial conditions of the system are assumed to be zero.
- The transfer function is a property of a system itself, independent of the magnitude and nature of the input or driving function.

传函的不同形式及相关概念开

$$G(s)=\frac{C(s)}{R(s)}=\frac{b_0 s^m+\cdots+b_{m-1}s+b_m}{a_0 s^n+a_1 s^{n-1}+\ldots+a_{n-1}s+a_n}=\frac{M(s)}{N(s)}$$

Polynomial TF
多项式传函

$$G(s)=\frac{b_0(s+z_1)(s+z_2)\cdots(s+z_m)}{a_0(s+p_1)(s+p_2)\cdots(s+p_n)}=K^*\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}$$

ZPK TF
零极点增益传函

$K^*=\dfrac{b_0}{a_0}$,根轨迹增益

$$G(s)=\frac{K(\tau_1 s+1)(\tau_2 s+1)\cdots(\tau_m s+1)}{(T_1 s+1)(T_2 s+1)\cdots(T_n s+1)}$$

时间常数传函，多用于频域分析

$K=\dfrac{b_m}{a_n}=K^*\cdot\dfrac{\prod_{i=1}^{m}(+z_i)}{\prod_{j=1}^{m}(+p_j)}$,开环增益

令G(s)中s=0,可得开环增益

Poles and zeros of TF

$$G(s)=\frac{M(s)}{N(s)}=\frac{b_0(s-z_1)(s-z_2)\cdots(s-z_m)}{a_0(s-p_1)(s-p_2)\cdots(s-p_n)}=K^*\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{j=1}^{n}(s-p_j)}$$

增益零极点（gain-zero-pole)传函

$s=z_i(i=1,2\cdots m)$ are called zeros of G(s)

$s=p_i(i=1,2\cdots n)$ are called poles of G(s)

Note: Since there exists somehow inertia in system component, n>=m. If m>n,the system is called unachievable.

How poles and zeros relate to system response

传递函数零极点与系统输出运动模态的关系

- Why we strive to obtain mathematical models?
- Why control engineers prefer to use TF model?
- How to use TF model to analyze and design control systems?

- we start from the relationship between the locations of zeros and poles of TF and the output responses of a system

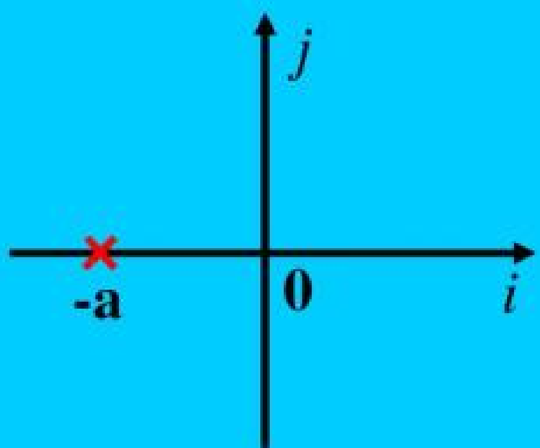
Transfer function

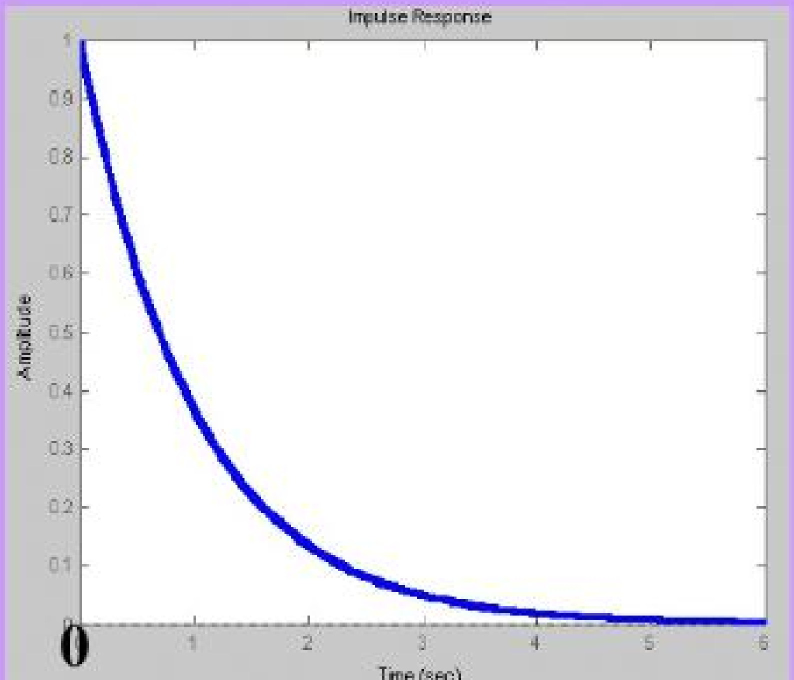
$$X(s) = \frac{A}{s+a}$$

Time-domain impulse response

$$x(t) = Ae^{-at}$$

Position of poles and zeros





Q: Where is the zero?

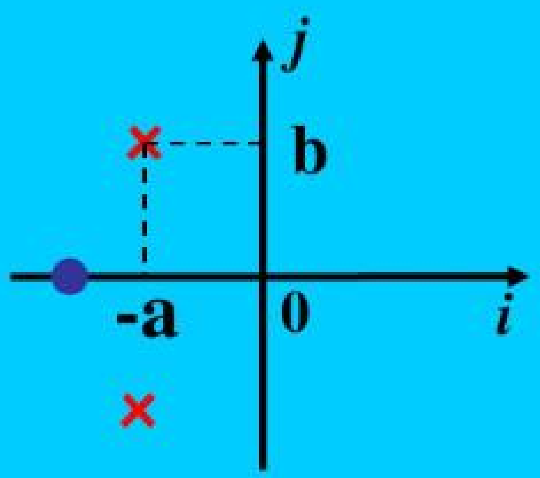
Transfer function

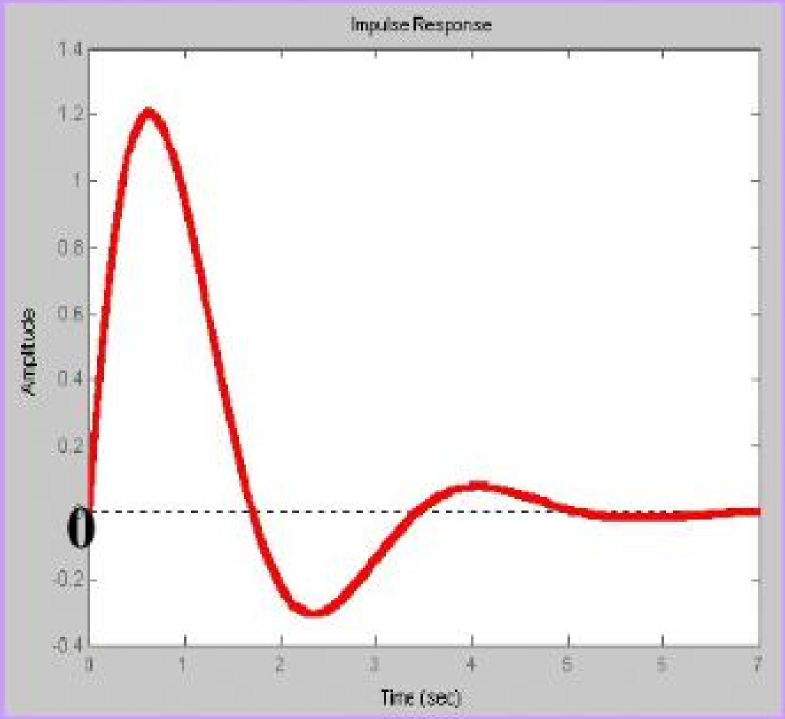
$$X(s) = \frac{A_1 s + B_1}{(s+a)^2 + b^2}$$

Time-domain impulse response

$$x(t) = Ae^{-at}\sin(bt+\phi)$$

Position of poles and zeros





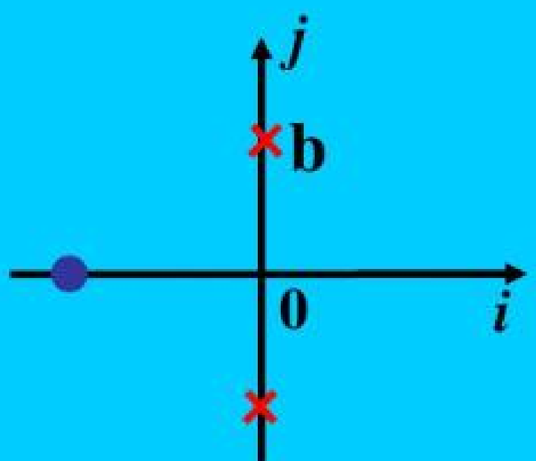
Transfer function

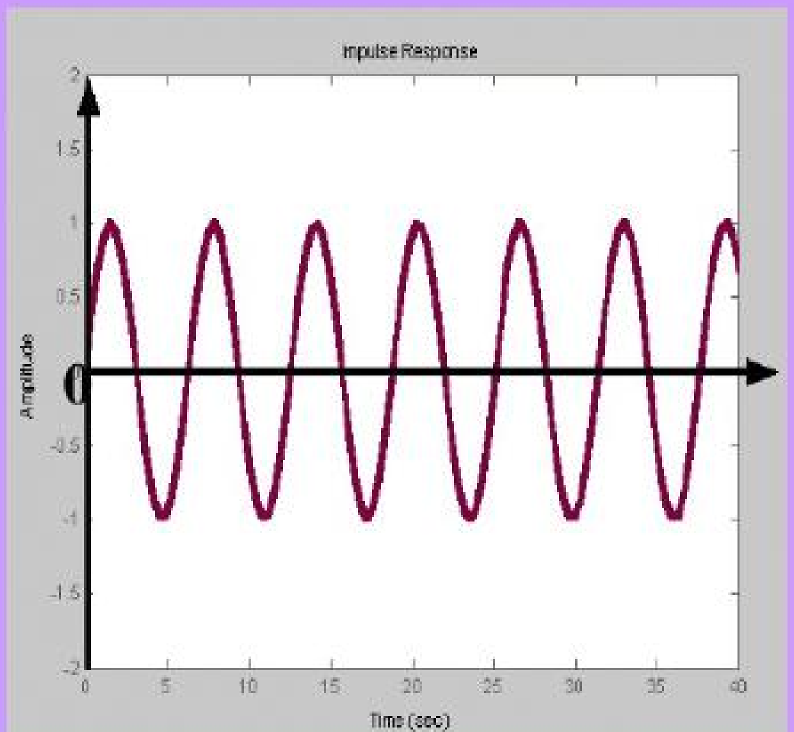
$$X(s) = \frac{A_1 s + B_1}{s^2 + b^2}$$

Time-domain impulse response

$$x(t) = A\sin(bt + \phi)$$

Position of poles and zeros





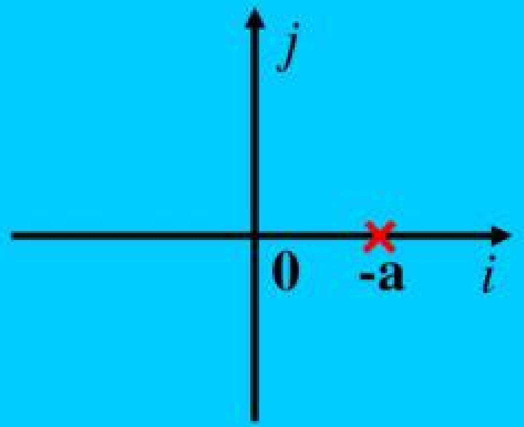
Transfer function

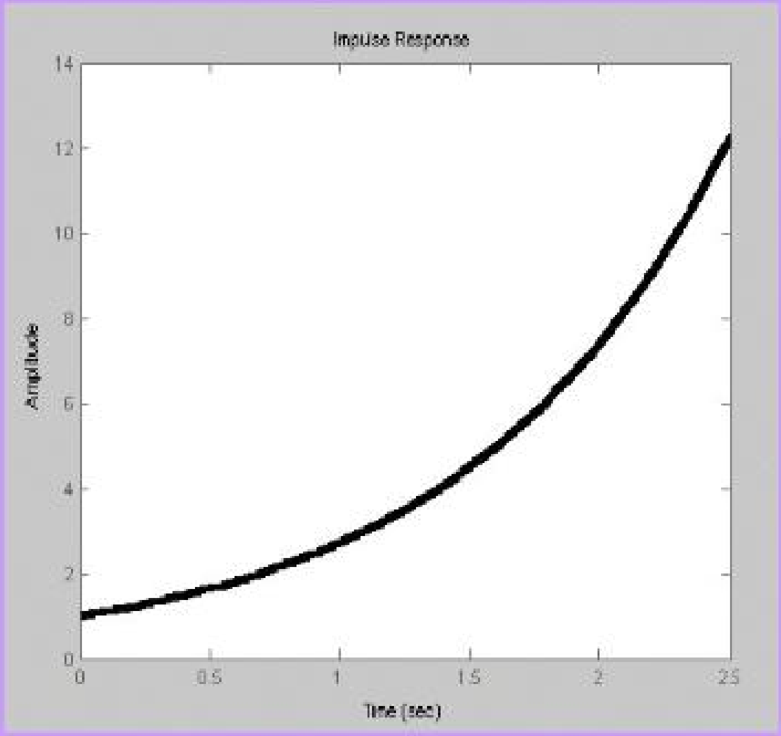
$$X(s) = \frac{A}{s-a}$$

Time-domain impulse response

$$x(t) = Ae^{at}$$

Position of poles and zeros





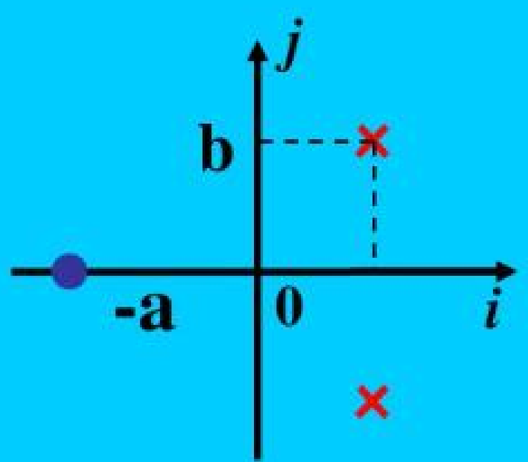
TF:

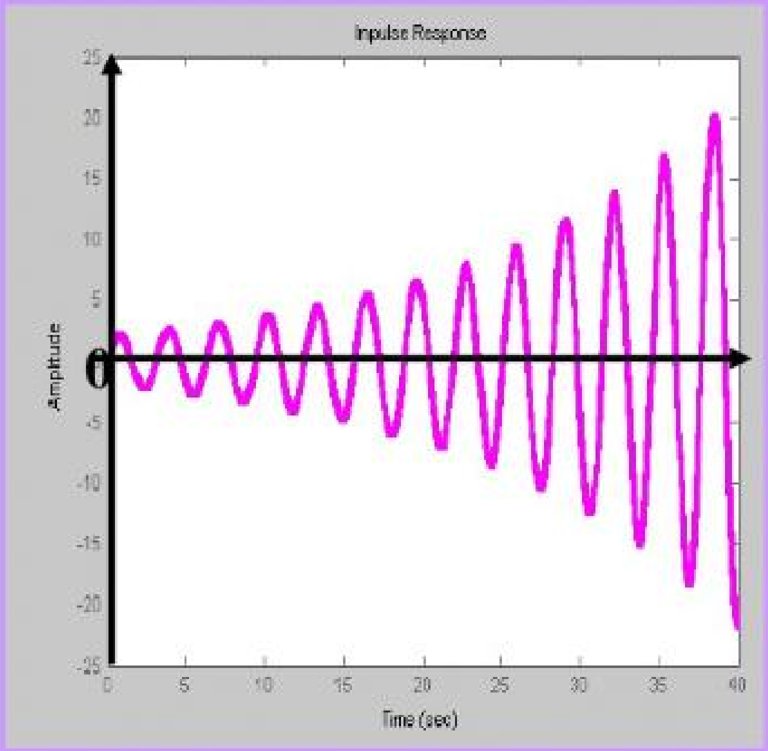
$$X(s)=\frac{A_1 s+B_1}{(s-a)^2+b^2}$$

Time-domain impulse response

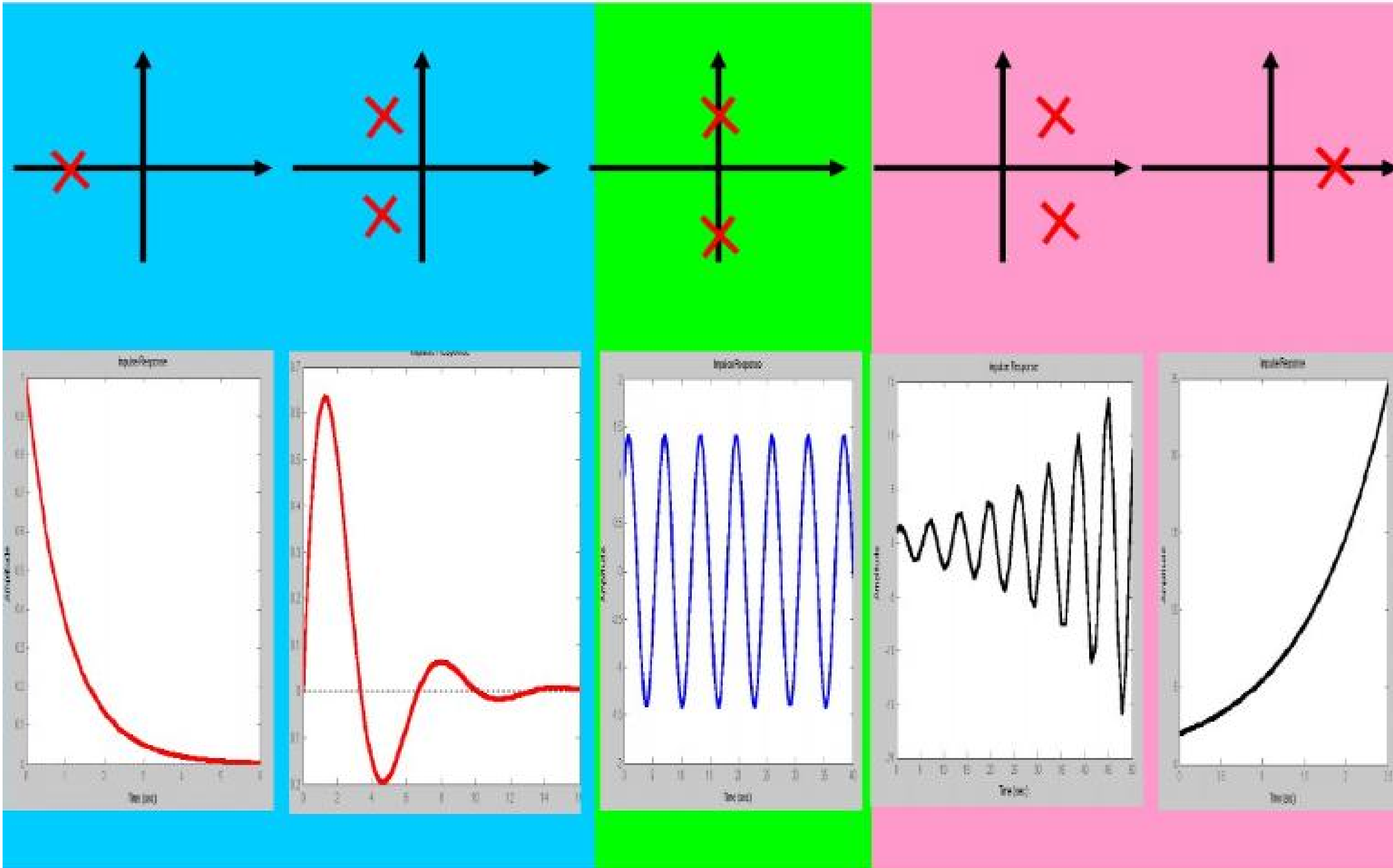
$$x(t)=Ae^{at}\sin(bt+\phi)$$

Position of poles and zeros





A summary of pole position and system dynamic response



Transfer function of typical component

1. Amplifying factor:

$$G(s) = K$$

2. Inertial factor :

$$G(s) = \frac{1}{Ts+1}$$

Transfer function of typical component

3. differential factor

$$G(s) = Ks$$

4. First order numerator term

$$G(s) = \tau s + 1$$

5. Second order numerator term

$$G(s) = \tau^2 s + 2\xi\tau s + 1$$

Transfer function of typical component

6. integral factor

$$G(s) = \frac{1}{s}$$

7. oscillatory factor

$$G(s) = \frac{\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n}$$

8. Time delay term

$$G(s) = e^{-\tau s}$$

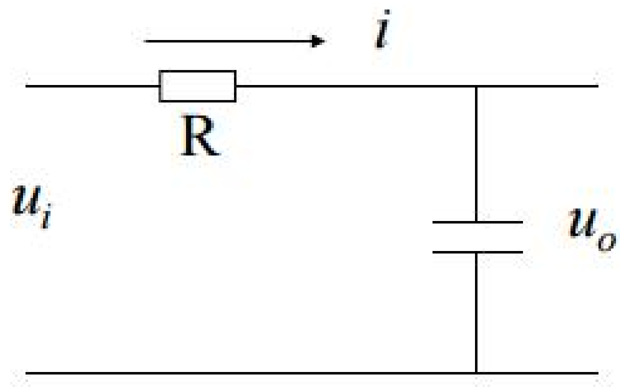
Transfer function of typical components – 2

元件	微分方程	传递函数
电桥 **Regulation resistance**	$u_\varepsilon(t) = k_\varepsilon \theta_\varepsilon(t)$	$G(s) = \dfrac{u_\varepsilon(s)}{\theta_\varepsilon(s)} = k_\varepsilon$
放大器 **Amplifier**	$u_a(t) = k_a u(t)$	$G(s) = \dfrac{u_a(s)}{u(s)} = k_a$
测速机 **Tachometer**	$u_t(t) = k_t \dfrac{d\theta_m(t)}{dt}$	$G(s) = \dfrac{u_t(s)}{\theta_m(s)} = k_t s$
电机 **DC motor**	$T_m \dfrac{d^2\theta_m(t)}{dt^2} + \dfrac{d\theta_m(t)}{dt} = k_m u_a(t)$	$G(s) = \dfrac{\theta_m(s)}{u_a(s)} = \dfrac{k_m}{(T_m s^2 + s)}$
减速器 **Reducer**	$\theta_c(t) = \dfrac{1}{i}\theta_m(t)$	$G(s) = \dfrac{\theta_c(s)}{\theta_m(s)} = \dfrac{1}{i}$

Transfer function of typical component

Example RC circuit obtain the uo(t)

$$u_i(t)=U_i \bullet 1(t), u_o(t)=0$$



Solution. Differential equation:

$$RC\frac{du_o(t)}{dt}+u_o(t)=u_i(t)$$

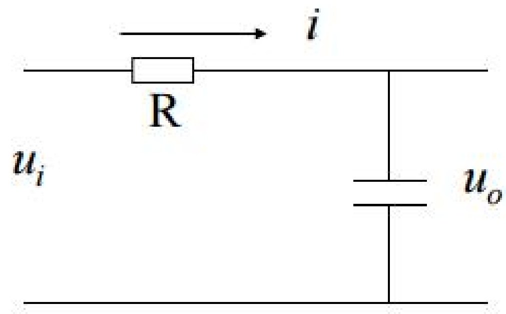
- Taking the Laplace transform of the both sides:

$$RCsU_o(s)+U_o(s)=U_i(s)$$

- Transfer function

$$G(s)=\frac{U_o(s)}{U_i(s)}=\frac{1}{RCs+1}$$

Transfer function of typical co



Example RC circuit obtain the uo(t)

$$u_i(t)=U_i \bullet 1(t), u_o(t)=0$$

Solution.

- Transfer function $G(s)=\dfrac{U_o(s)}{U_i(s)}=\dfrac{1}{RCs+1}$
- Solving $U_o(s)$,we have

$$U_o(s)=\frac{U_i}{s\left(RCs+1\right)}=\frac{U_i}{s}-\frac{U_i}{s+\dfrac{1}{RC}}$$

- Taking the inverse LT:

$$u_o(\mathrm{t})=U_i-U_i e^{-\frac{t}{RC}}=U_i(1-e^{-\frac{t}{RC}})$$

Exercise1: The following differential equations represent linear time-invariant systems, where r(t) denotes the input and c(t) denotes the output. Find the transfer function of each of the systems.

(a) $$\frac{d^3c(t)}{dt^3}+6\frac{d^2c(t)}{dt^2}+11\frac{dc(t)}{dt}+6c(t)=r(t)$$

(b) $$\frac{d^3c(t)}{dt^3}+3\frac{d^2c(t)}{dt^2}+4\frac{dc(t)}{dt}+c(t)=2\frac{dr(t)}{dt}+r(t)$$

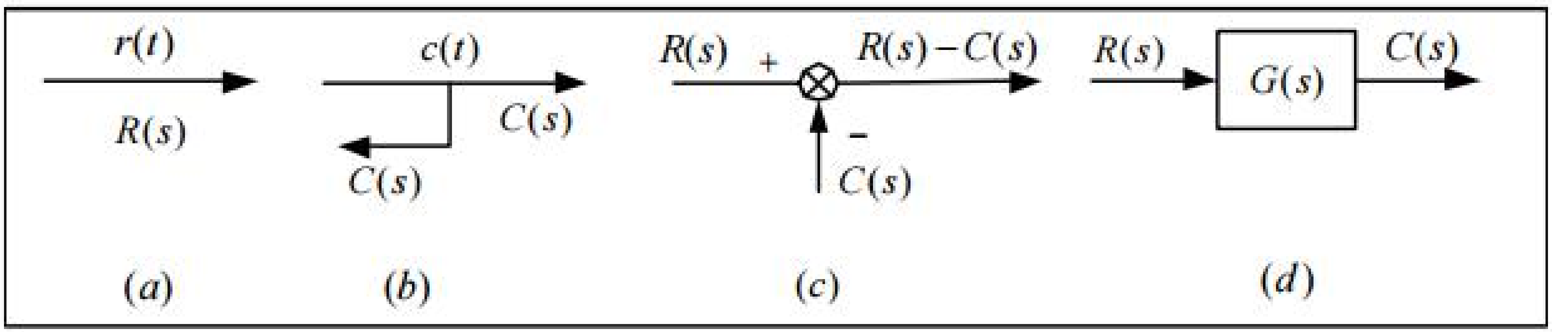
Block diagram

- If the differential equations are known, by taking the Laplace transform of sub-equation of equations and drawing the corresponding sub-block diagram, the block diagram of the system can be obtained by connecting the sub-block diagram; if the structure diagram is known, then transfer the name of every component into its corresponding transfer function and represent all the variables by the Laplace form, thus the block diagram can be obtained.

Block diagram

Composition

Function block: input signal, output signal, transfer function.



(a) signal line;
(b) pickoff point;
(c) summing junction;
(d) block;

Block diagram

Example. Obtain the block diagram

$$
\begin{cases}
u_a = L_a \dfrac{di_a(t)}{dt} + R_a i_a(t) + E_a \\
E_a = C_e \omega_m(t) \\
M_m(t) = C_m(t) i_a(t) \\
J_m \dfrac{d\omega_m(t)}{dt} + f_m \omega_m(t) = M_m(t) - M_c(t)
\end{cases}
$$

Block diagram

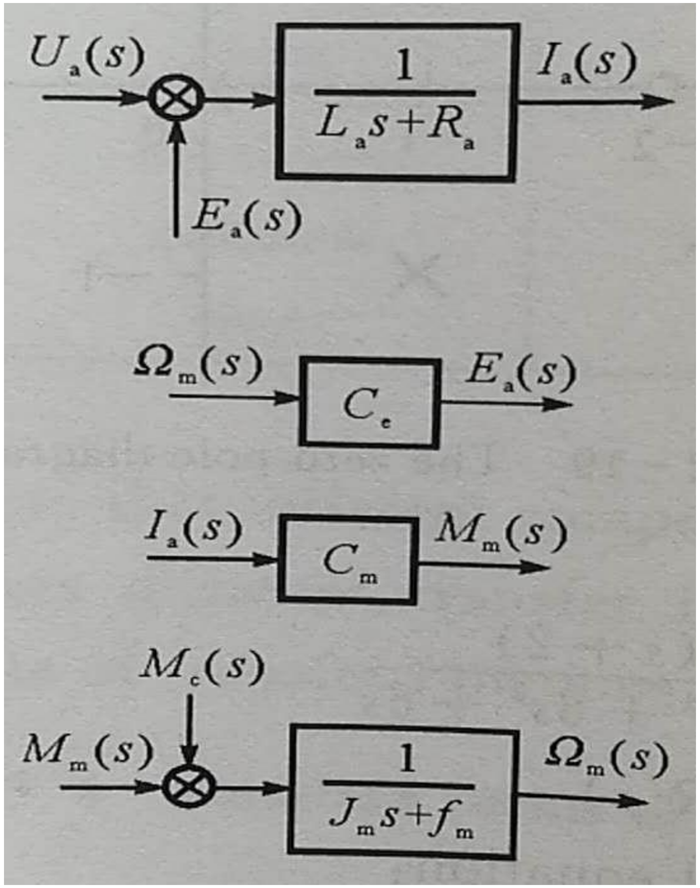
Example. Obtain the block diagram

$$
\begin{cases}
\dfrac{I_a(s)}{U_a(s)-E_a(s)} = \dfrac{1}{L_a s + R_a} \\[2ex]
\dfrac{E_a(s)}{\Omega_m(s)} = C_e \\[2ex]
\dfrac{M_m(\mathrm{s})}{I_a(s)} = C_m \\[2ex]
\dfrac{\Omega_m(s)}{M_m(\mathrm{s})-M_c(s)} = \dfrac{1}{J_m s + f_m}
\end{cases}
$$

Block diagram

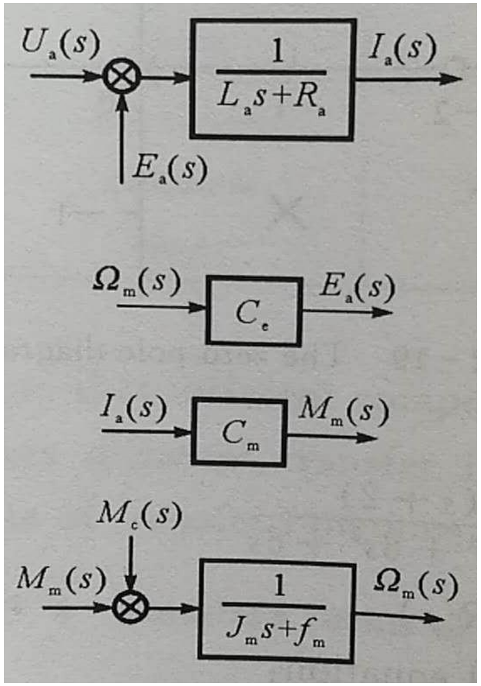
Example. Obtain the block diagram

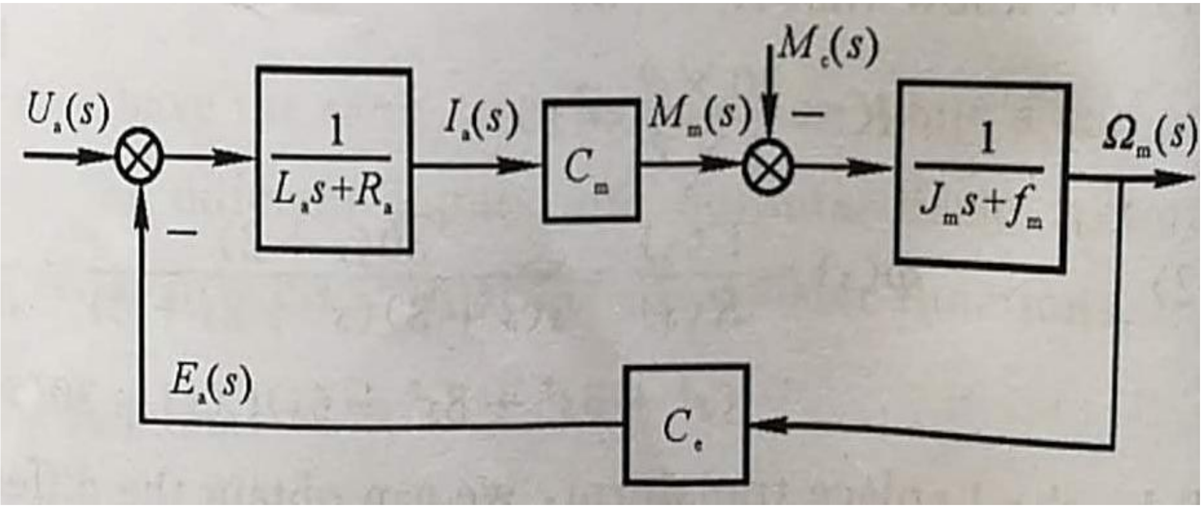
$$
\begin{cases}
\dfrac{I_a(s)}{U_a(s)-E_a(s)}=\dfrac{1}{L_a s+R_a} \\
\dfrac{E_a(s)}{\Omega_m(s)}=C_e \\
\dfrac{M_m(\mathrm{s})}{I_a(s)}=C_m \\
\dfrac{\Omega_m(s)}{M_m(\mathrm{s})-M_c(s)}=\dfrac{1}{J_m s+f_m}
\end{cases}
$$



Block diagram

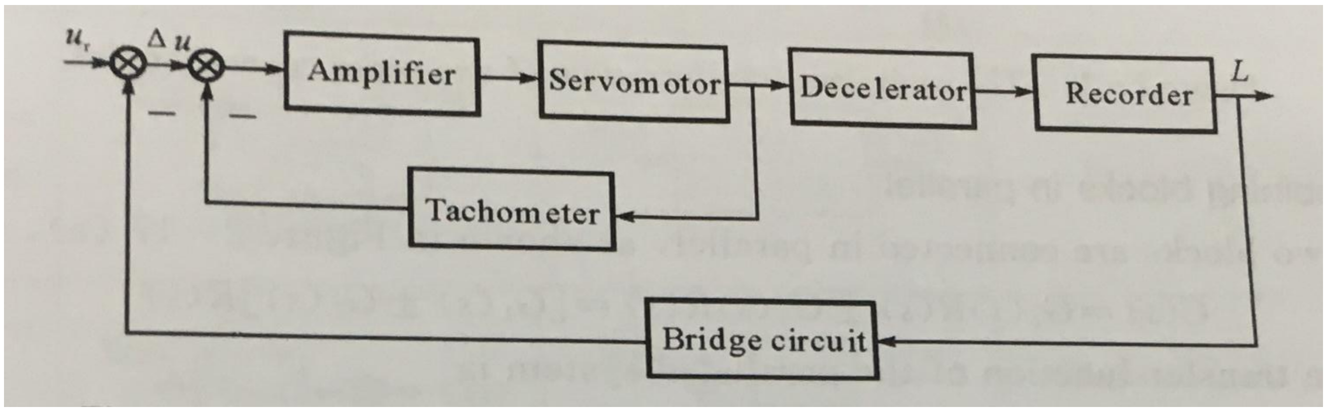
Example. Obtain the block diagram

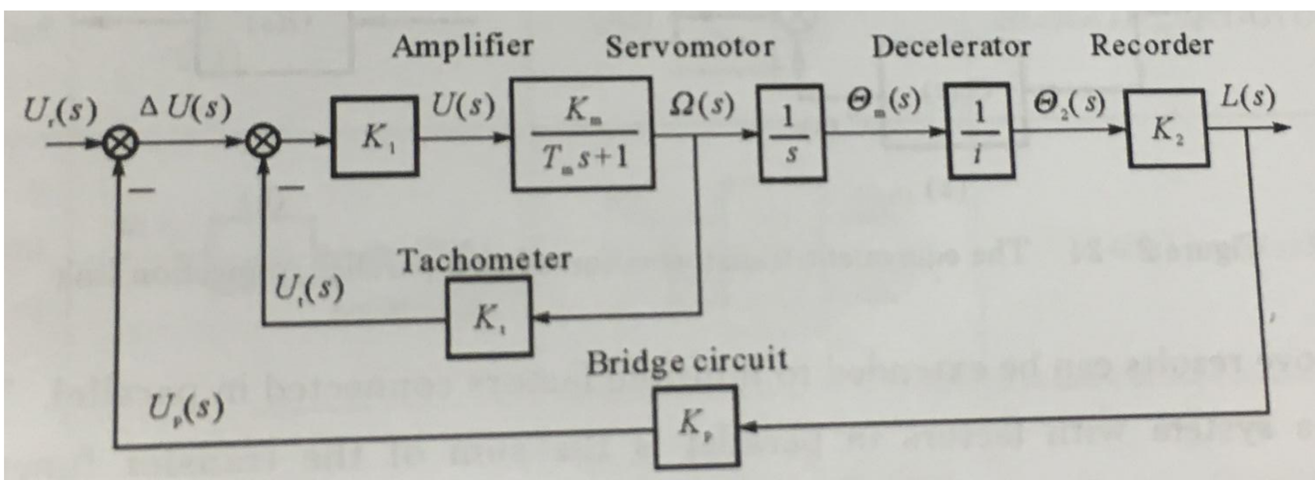




Block diagram

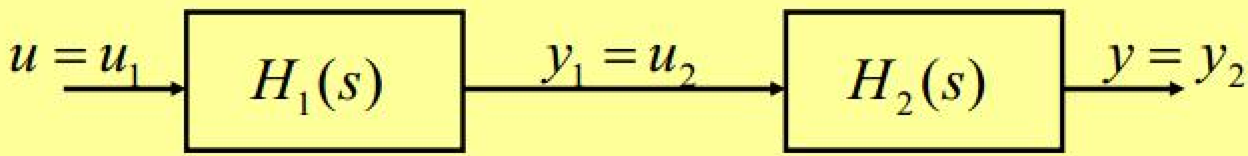
Example. Obtain the corresponding block diagram for the structure diagram.





Simplification of block diagram

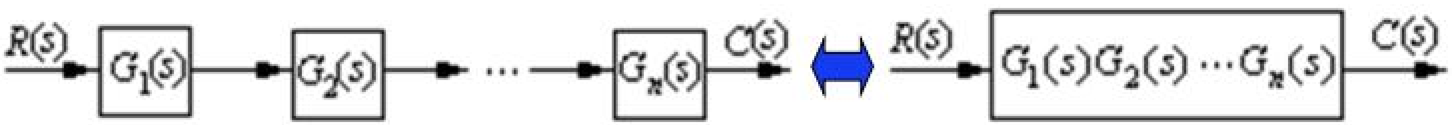
1. Serial blocks



$$Y(s) = Y_2(s) = H_2(s)U_2(s) = H_2(s)Y_1(s) = H_2(s)H_1(s)U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = H_2(s)H_1(s)$$

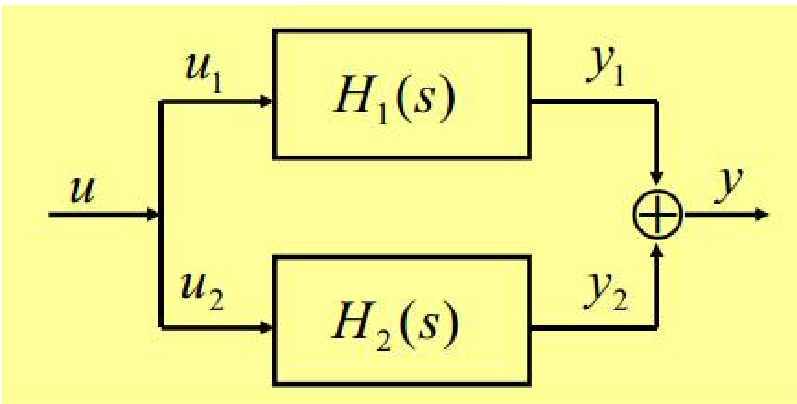
N blocks are in series



(a) (b)

Simplification of block diagram

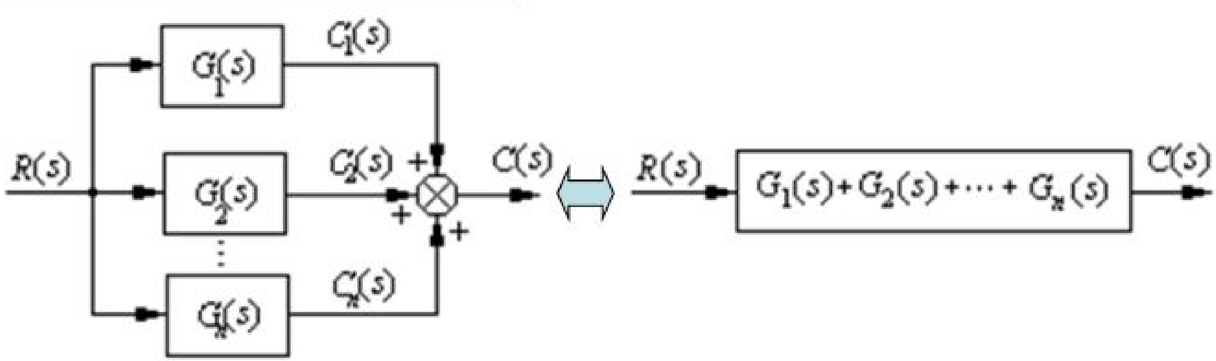
2. Parallel blocks



$$G(s) = \frac{Y(s)}{U(s)} = H_1(s) + H_2(s)$$

$$Y(s) = Y_1(s) + Y_2(s) = H_1(s)U_1(s) + H_2(s)U_2(s) = \left(H_1(s) + H_2(s)\right)U(s)$$

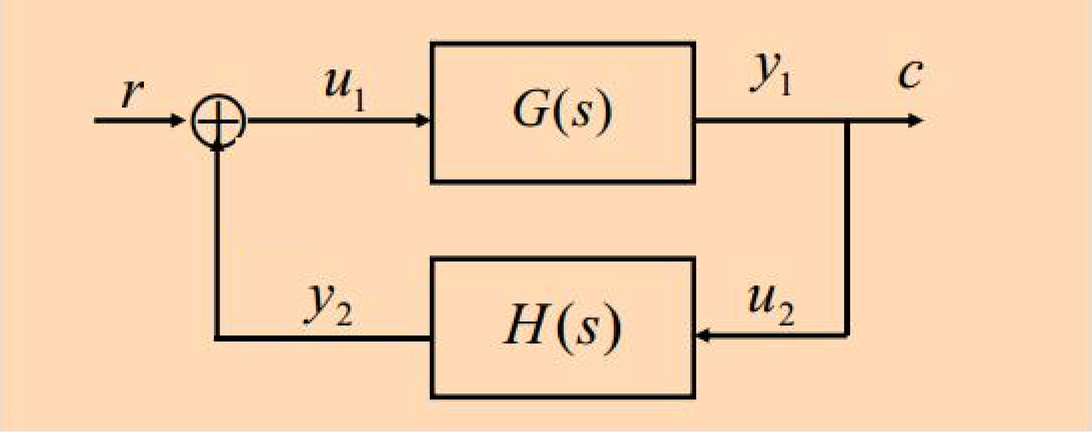
N blocks are in parallel



(a) (b)

Simplification of block diagram

3. Feedback



$$C(s)=G(s)U_1(s)=G(s)[R(s)+Y_2(s)]=G(s)[R(s)+H(s)C(s)]$$

$$C(s)[1-G(s)H(s)]=G(s)R(s)$$

$$\Phi(s)=\frac{C(s)}{R(s)}=\frac{G(s)}{1-G(s)H(s)} \Rightarrow \Phi(s)=\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}$$

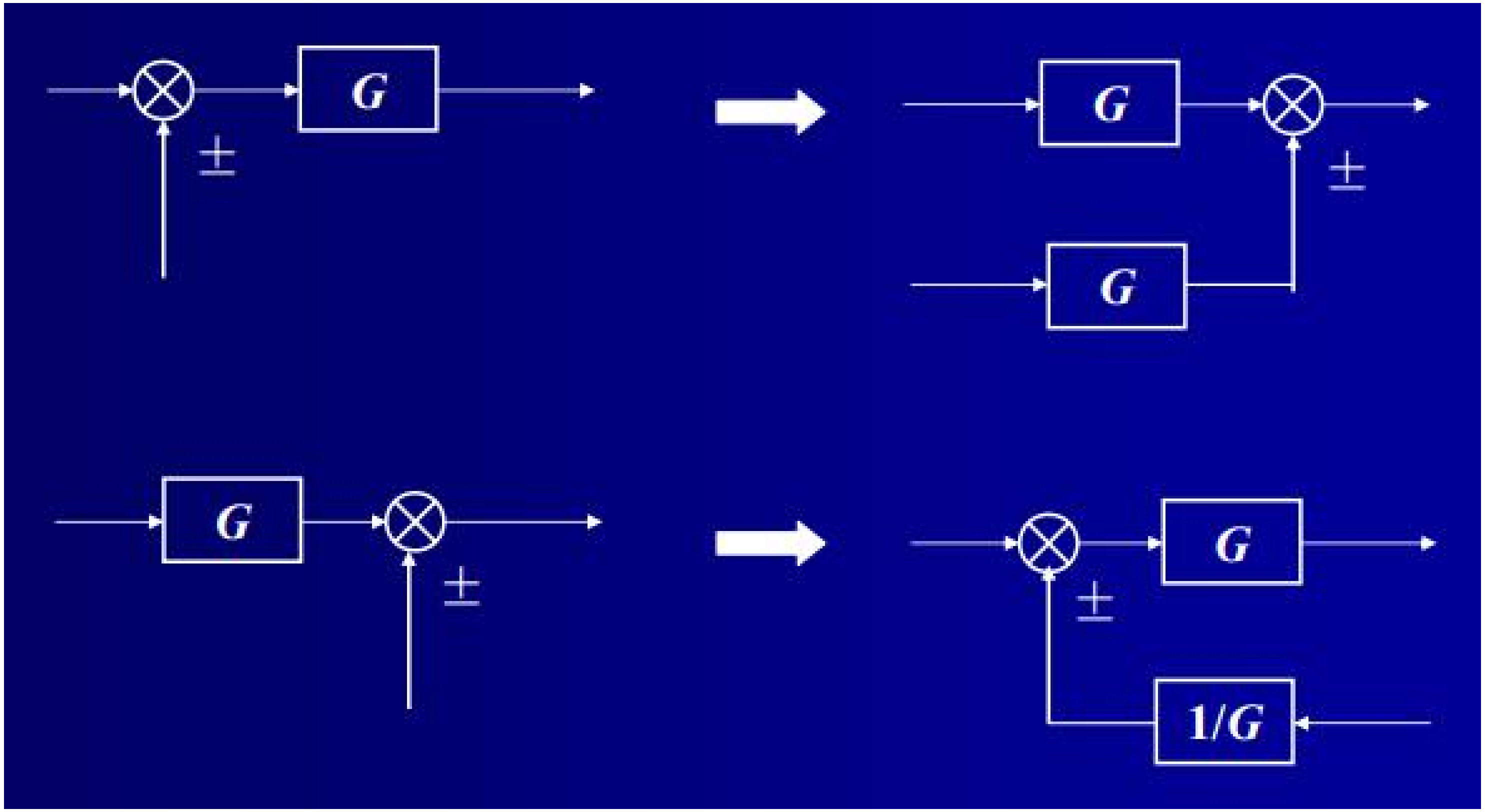
positive feedback

Note!

negative feedback

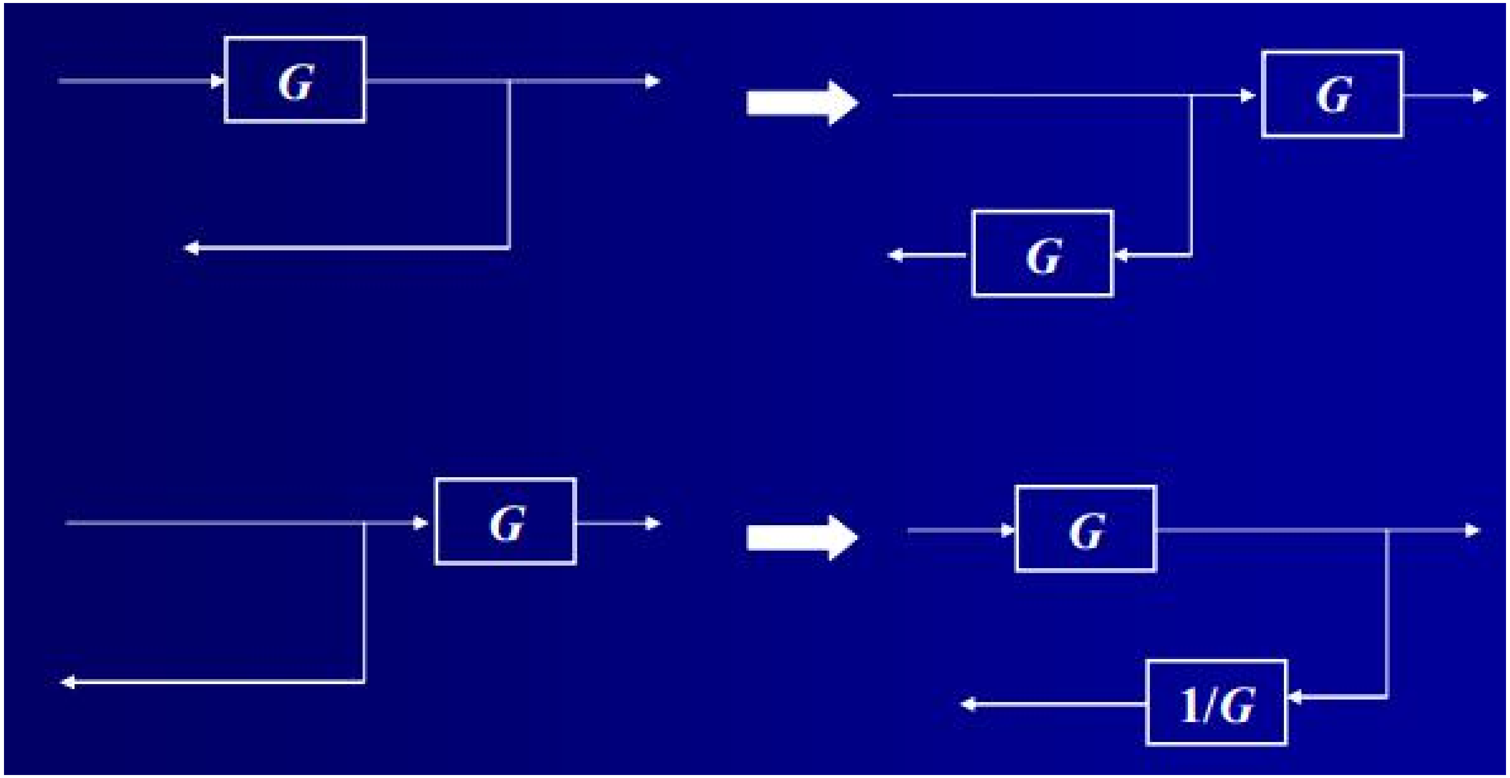
Simplification of block diagram

Moving a summing junction



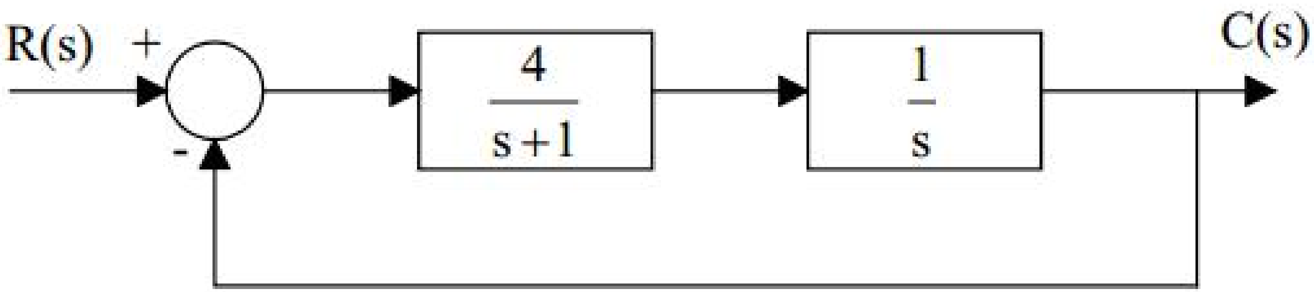
Simplification of block diagram

Moving a pickoff point



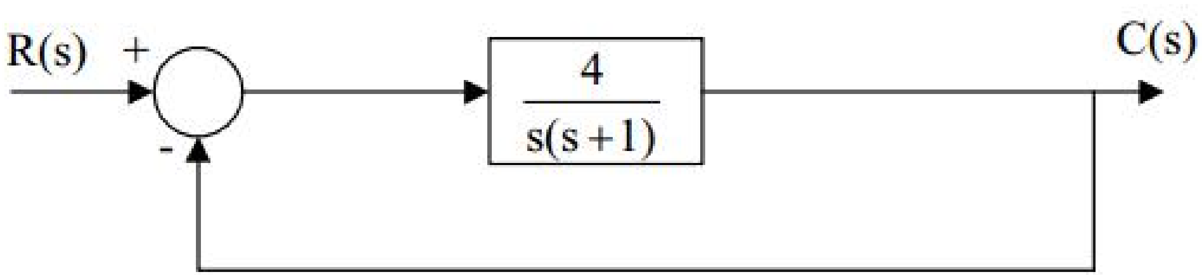
Simplification of block diagram

Example 1: Find the transfer function of the closed-loop system below



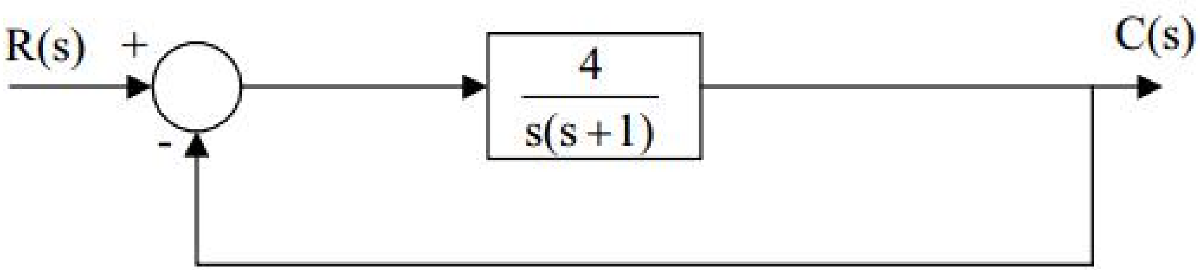
Solution:

Use case 1 to combine the two serials blocks



Simplification of block diagram

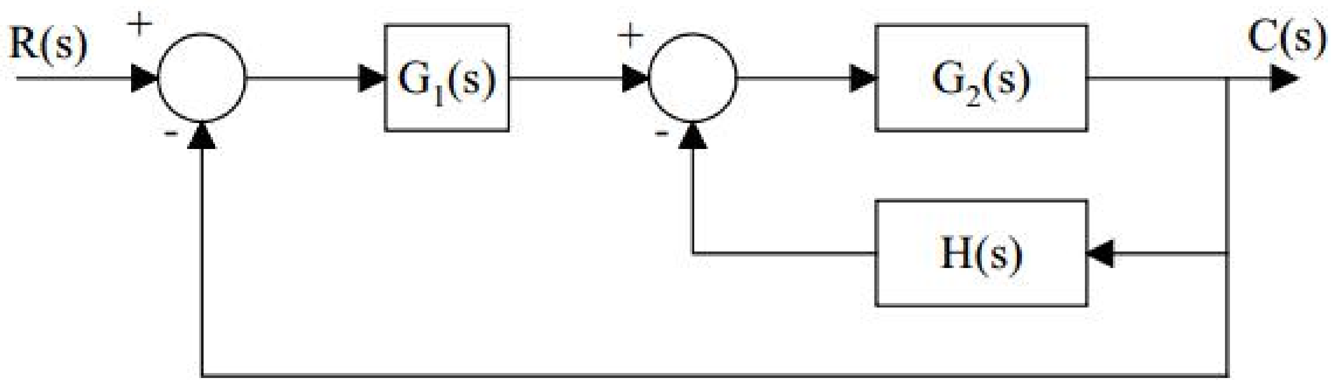
Example 1:



Use case 3 to obtain the transfer function

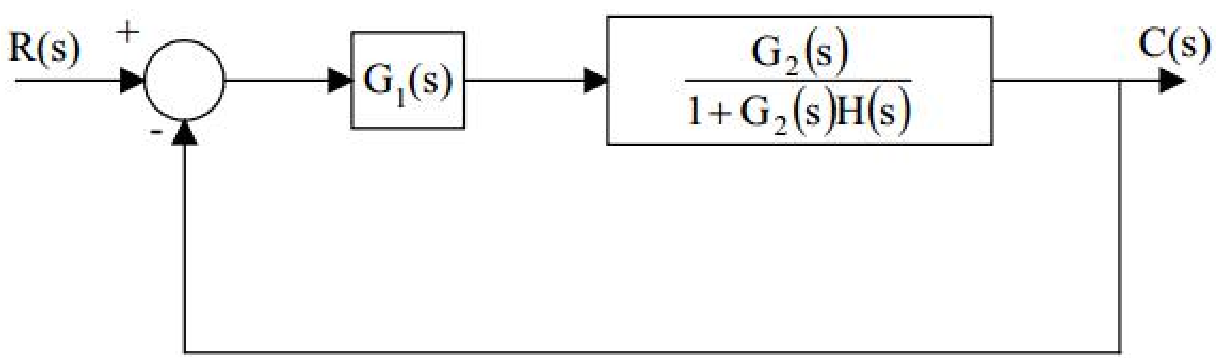
$$\frac{C(s)}{R(s)}=\frac{\dfrac{4}{s(s+1)}}{1+\dfrac{4}{s(s+1)}\cdot 1}=\frac{4}{s^2+s+4}$$

Example 2: Find the transfer function of the closed-loop system below

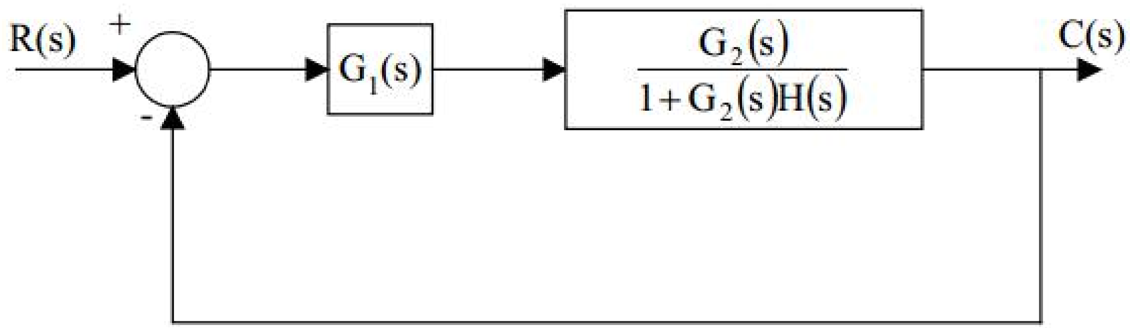


Solution:

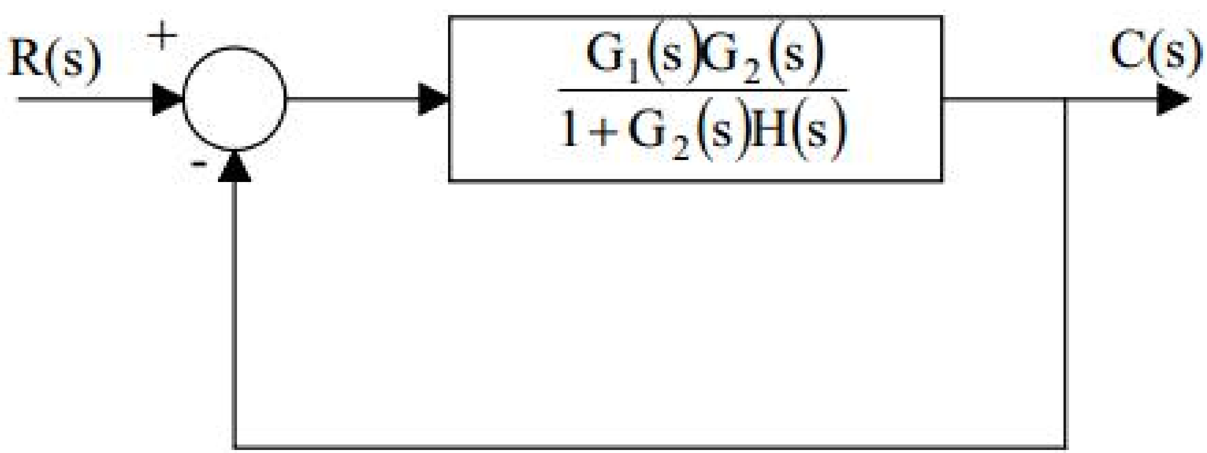
Use case 3 to simplify the inner feedback loop



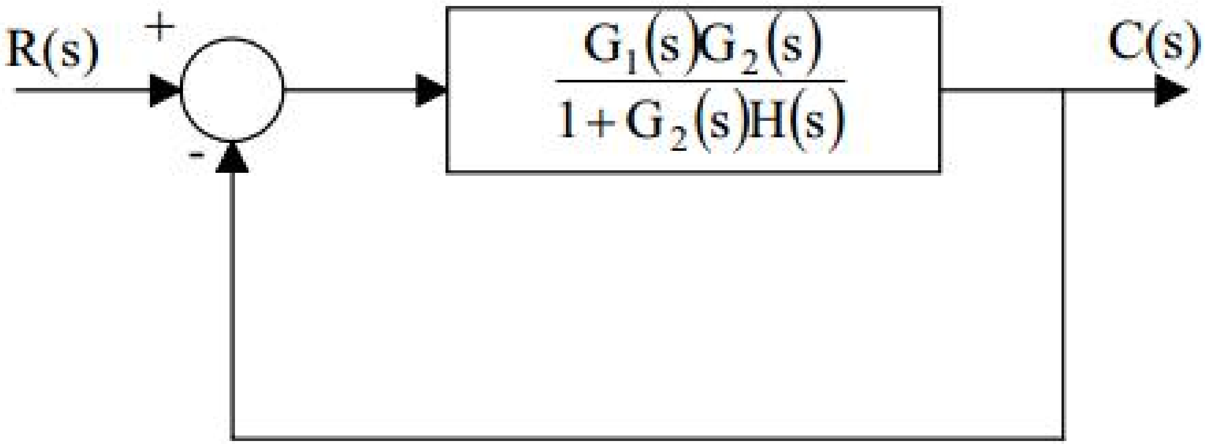
Example 2



Use case 1 to combine two series block into one.



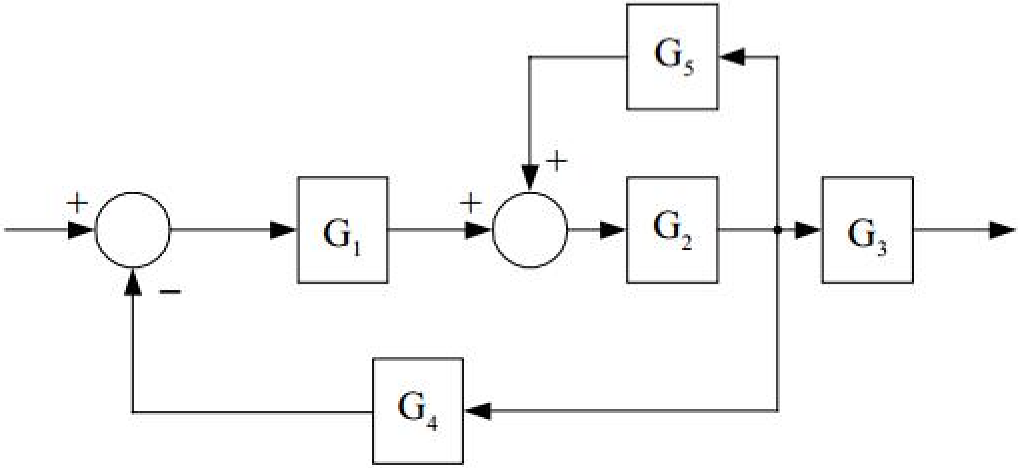
Example 2



Use case 3 to obtain the transfer function

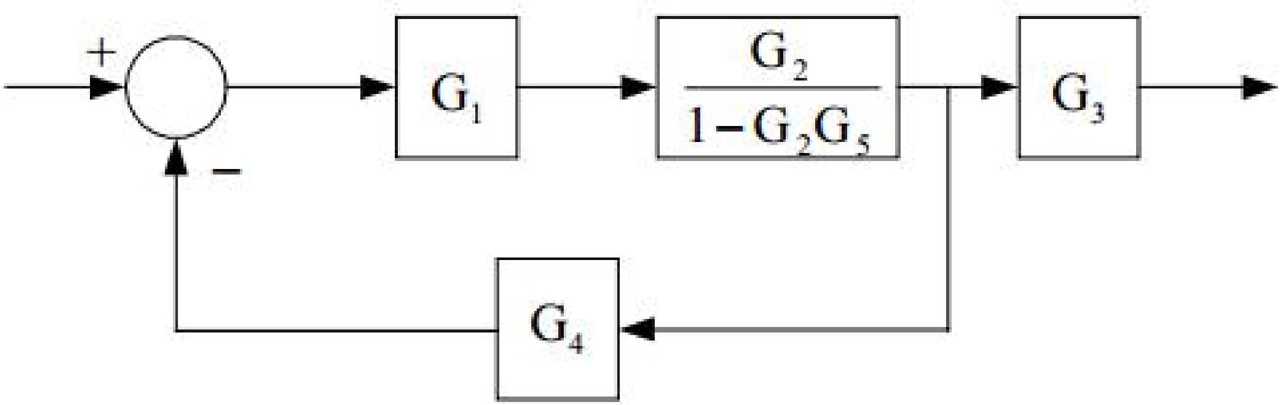
$$\frac{C(s)}{R(s)} = \frac{\dfrac{G_1(s)G_2(s)}{1+G_2(s)H(s)}}{1+\dfrac{G_1(s)G_2(s)}{1+G_2(s)H(s)}} = \frac{G_1(s)G_2(s)}{1+G_2(s)H(s)+G_1(s)G_2(s)}$$

Example 3: Find the transfer function

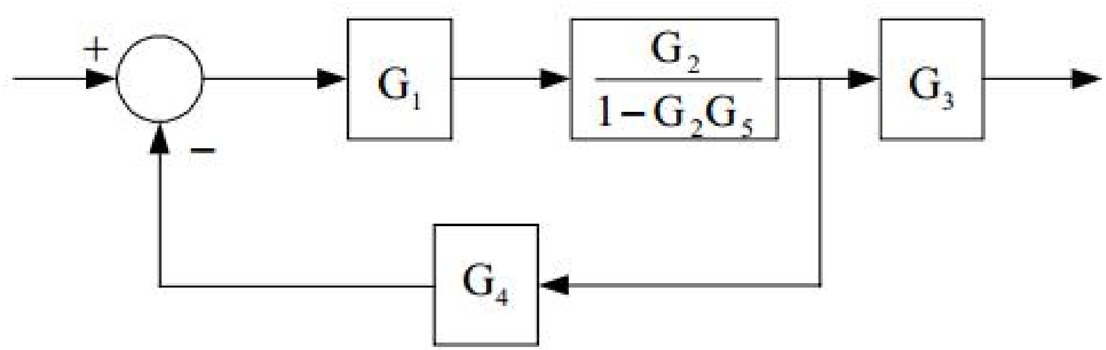


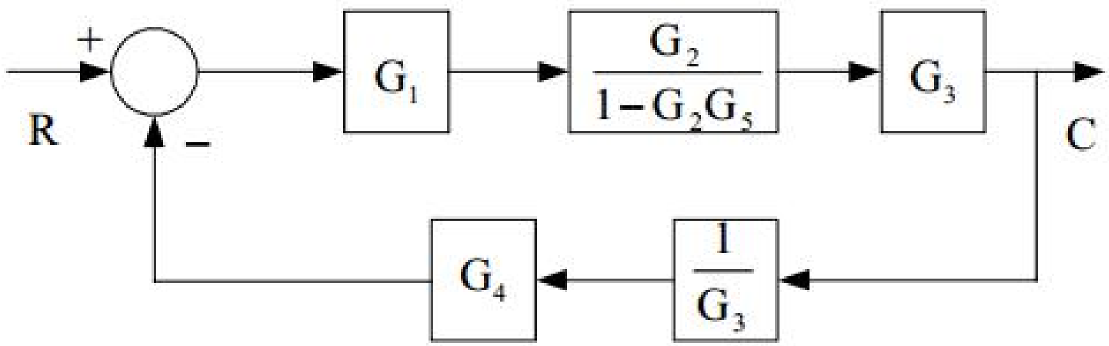
Solution:

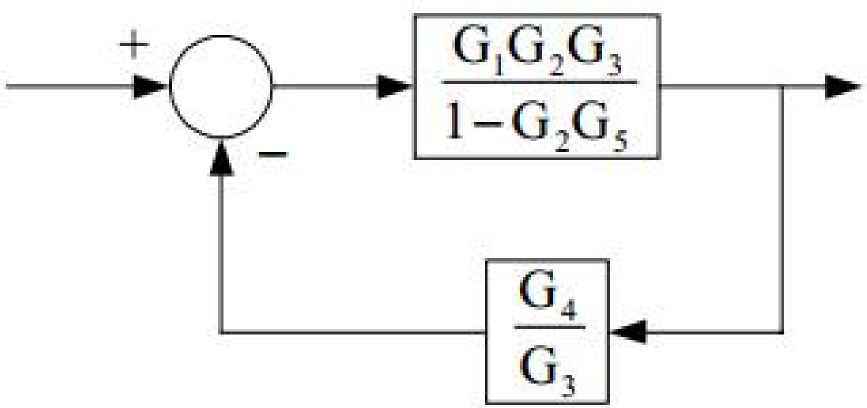
Use case 3 to simplify the inner feedback blocks



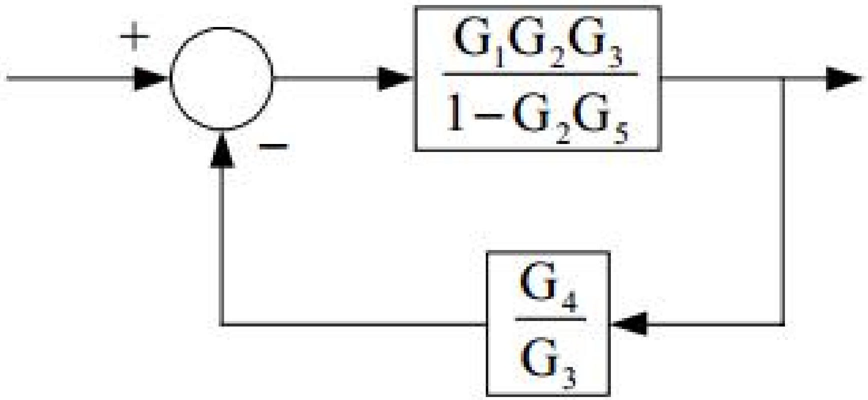
Example 3:


+
−
G_1
$\frac{G_2}{1-G_2G_5}$
G_3
G_4


+
R
−
G_1
$\frac{G_2}{1-G_2G_5}$
G_3
C
G_4
$\frac{1}{G_3}$


+
−
$\frac{G_1G_2G_3}{1-G_2G_5}$
$\frac{G_4}{G_3}$

Example 3:



Calculate the overall transfer function

$$\frac{C}{R}=\frac{\dfrac{G_1G_2G_3}{1-G_2G_5}}{1+\dfrac{G_1G_2G_3}{1-G_2G_5}\cdot\dfrac{G_4}{G_3}}=\frac{G_1G_2G_3}{1-G_2G_5+G_1G_2G_4}$$