Principles of Automatic Control

CHAPTER 4

Stability Analysis of Linear Systems

Outline

The concept of stability

-Definition

When a system is offset its equilibrium by a disturbance, the system is stable if it can return to the original equilibrium with sufficient accuracy. Otherwise, it is unstable.

Note 1: For linear systems, stability means asymptotic stability, but it is not true for nonlinear systems.

Note 2: Stability of linear systems depends only on internal structure of the system, independent of external action and initial condition.

Sufficient and necessary condition（充分必要条件） for stability

$$\phi(s)=\frac{C(s)}{R(s)}=\frac{b_0s^m+b_1s^{m-1}+\cdots+b_{m-1}s+b_m}{a_0s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n}=\frac{M(s)}{D(s)}$$

$$M(s)=b_0s^m+b_1s^{m-1}+\cdots+b_{m-1}s+b_m$$

$$D(s)=a_0s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n$$

The system output is $C(s)=\frac{M(s)}{D(s)}\cdot R(s)$

时域定义：

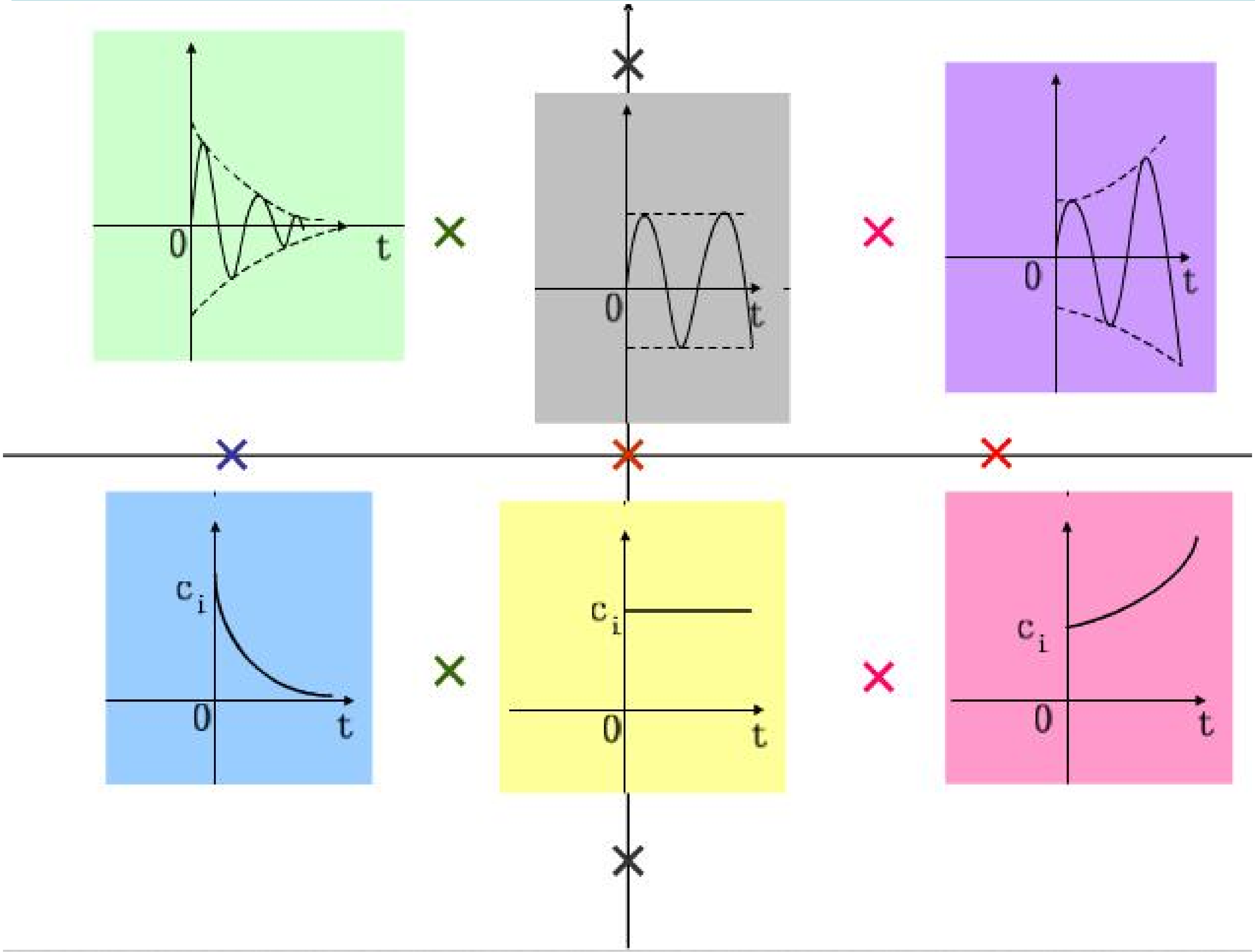
The initial condition of the system is zero. When system input is unit impulse functionδ(t), the system output is k(t).

If $\lim_{t\to\infty} k(t)=0$, then the system is stable。

$$C(s)=\frac{M(s)}{D(s)}=\sum_{i=1}^{n}\frac{C_i}{s-s_i} \qquad k(t)=c(t)=\sum_{i=1}^{n} C_i e^{s_i t}$$

$$\lim_{t\to\infty} k(t)=0 \quad \Rightarrow \quad e^{s_i t} \text{ decay with time}$$

All poles should locate in the left side of s-plane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Stability criterion in complex plane

A system is stable if and only if all roots of the system characteristic equation have negative real parts, or equivalently, all poles of closed-loop transfer functions must locate in the left half of s-plane.

Note: The above criterion holds when the characteristic equation has multiple-order roots。

Routh stability criterion

All poles in left s plane	=	No poles in right s plane

- The criterion tests whether any of the roots of the characteristic equation lie in the right half of the s-plane, without actually solving for the roots.
- Information about absolute stability can be obtained directly from the coefficients of the characteristic equation

Procedures in Routh's criterion

- The necessary condition: Consider that the characteristic equation of a LTI system is of the form

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0 \quad a_n > 0$$

The necessary (but not sufficient) conditions for stability of the system are described as follows:

(1) All the coefficients of the equation have the same sign. ($a_i > 0$)

(2) None of the coefficients vanishes.

Procedures in Routh's criterion

- Routh's Tabulation or Routh's Array

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$	$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$	b_3	b_4	$\cdots$
s^{n-3}	$c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1}$	$c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1}$	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
s^0	a_0				

- **Routh criterion:**
- The system is stable if and only if all the elements of the first column of the Routh's tabulation are of the same sign.(表中第一列系数均为正数)
- The number of changes of signs in the elements of the first column equals the number of roots with positive real parts or in the right half of the s-plane. (符号改变次数等于右半平面根的个数)

- Example 1 $D(s)=s^4+2s^3+3s^2+4s+5=0$

- Solution:

- In the first column there are two changes of sign, therefore the equation has two characteristic roots in the right-half s plane. The system is unstable.

Special cases of Routh' array

- Case 1: The first element in any row of Routh's array is zero.

When the first element in a row is zero but not all the other elements are zeros. To remedy the situation, we replace the zero element in the first column by an arbitrary small positive number of ε, and then proceed with Routh's array.

Special cases of Routh' array

- Case 1: The first element in any row of Routh's array is zero.

$$s^4+3s^3+4s^2+12s+16=0$$

s^4	1	4	16
s^3	3	12	
s^2	$0(\varepsilon)$	16	
s^1	$\dfrac{12\varepsilon-48}{\varepsilon}$	0	
s^0	16		

当ε很小时，

$$\frac{12\varepsilon-48}{\varepsilon}=12-\frac{48}{\varepsilon}<0$$

则系统不稳定，并有两个正实部根。

10:

Special cases of Routh' array

- Case 2: All elements are zeros.

When the elements in one row of Routh's array are all zero, the situation can be remedied by using the auxiliary equation, which is formed from the coefficients of the row just above the row of zeros in Routh's array. Replace the row of zeros with the coefficients of the auxiliary equation.

On-class exercises

Determine the stability of the following systems:

$$(1)\ 2s^4 + s^3 + 3s^2 + 5s + 10 = 0$$

$$(2)\ s^4 + s^3 + 2s^2 + 2s + 3 = 0$$